Chapter 2: First Order Differential Equations

Section 2-8: Equilibrium Solutions (Logistic Equation)

An ordinary differential equation in which the independent variable does not appear explicitly is said to be **autonomous**. That is, an autonomous first order differential equation has the form

$$\frac{dy}{dx} = f(y). \tag{7}$$

Example 23. Determine whether the following first order differential equations are autonomous or nonautonomous.

(a) $\frac{dy}{dx} = 3y^{2/3}$ $f(y) = 3y^{2/3}$: autonomous

(b) $\frac{dy}{dx} = y(3-y)$ $f(y) = y(3-y)$: autonomous

(c) $\frac{dy}{dx} = -\frac{x}{y}$ $f(x,y) = -\frac{x}{y}$: nonautonomous

(d) $\frac{dy}{dx} = \frac{2x}{y+x^2y}$ $f(x,y) = \frac{2x}{y+x^2y}$: nonautonomous

Suppose we wish to model the population of some species over time. We could start with a simple model such as

$$\frac{dP}{dt} = rP,$$

where $P(t)$ is the population at time $t \geq 0$ and r is the growth rate. Provided the population is small, this model makes some sense. However, when the population is large, limiting factors (such as the availability of food and other resources, as well as space) will affect the growth of the population. Taking into account such limiting factors, it would make more sense to use a model like

$$\frac{dP}{dt} = g(P) \cdot P,$$

where the growth rate $g(P)$ has the following properties:

- $g(P) \approx r$ when the population is small,
- $g(P)$ decreases as population increases, and
- $g(P)$ is negative when the population is sufficiently large.

One such example is the so-called **logistic equation**

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P, \tag{8}$$

where $r > 0$ is the **intrinsic growth rate** (the growth rate that will occur in the absence of any limiting factors) and $K > 0$ is the **carrying capacity** (the maximum sustainable population).

Example 24. Solve Equation (8) with the initial condition $P(0) = P_0$.

From (8), using the method of separation of variables

$$\frac{dP}{\left(1 - \frac{P}{K}\right)P} = r\,dt$$

Integrating both sides

$$LHS = \int \frac{dP}{\left(1 - \frac{P}{K}\right)P} = \int r\,dt = RHS$$

$$RHS = \int r\,dt = rt + C_1$$

For LHS :

$$\frac{1}{\left(1-\frac{P}{K}\right)\cdot P} = \frac{A}{1-\frac{P}{K}} + \frac{B}{P} = \frac{AP + B\left(1-\frac{P}{K}\right)}{\left(1-\frac{P}{K}\right)\cdot P}$$

$$= \frac{P\left(A-\frac{B}{K}\right)+B}{\left(1-\frac{P}{K}\right)\cdot P} \Rightarrow B = 1$$

$$A - \frac{B}{K} = 0 \Rightarrow A = \frac{1}{K}$$

$$LHS = \int \frac{1}{\left(1-\frac{P}{K}\right)\cdot P}\, dP = \int \frac{1/K}{1-P/K}\, dP + \int \frac{1}{P}\, dP$$

$$= \int \frac{1}{K-P}\, dP + \ln P + C_2$$

$$= -\ln(K-P) + \ln P + C_2$$

$$= \ln\left(\frac{P}{K-P}\right) + C_2$$

$$\Rightarrow LHS = RHS \Rightarrow \ln\left(\frac{P}{K-P}\right) + C_2 = rt + C_1$$

$$\Rightarrow e^{\ln\left(\frac{P}{K-P}\right)} = e^{rt + C}, \quad C = C_1 - C_2$$

$$\Rightarrow \frac{P}{K-P} = Ce^{rt} \Rightarrow P = KCe^{rt} - PCe^{rt}$$

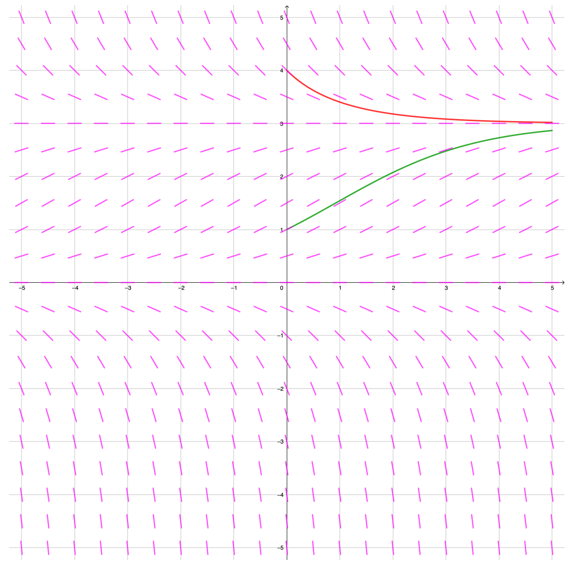
$$\Rightarrow P(1+Ce^{rt}) = KCe^{rt} \Rightarrow P = \frac{KCe^{rt}}{1+Ce^{rt}} \qquad (9)$$

Divide the numerator and the denominator by Ce^{rt}

$$\Rightarrow P(t) = \frac{K}{1+Ce^{-rt}} \quad \text{and} \quad P(0) = P_0 = \frac{K}{1+C}$$

$$\Rightarrow 1 + C = \frac{K}{P_0} \Rightarrow C = \frac{K}{P_0} - 1$$

Example 25. Below is the direction field corresponding to the logistic equation $\frac{dP}{dt} = 0.75\left(1 - \frac{P}{3}\right)P$, along with solutions when $P(0) = 1$ (green curve) and $P(0) = 4$ (red curve).



(a) What are the constant solutions to this logistic equation?

$$\frac{dP}{dt} = 0 \iff 0.75\left(1 - \frac{P}{3}\right) \cdot P \iff P = 3 \text{ or } P = 0$$

are constant (equilibrium solutions)

$K = 3$

(b) Is the solution increasing or decreasing when P is between 0 and the carrying capacity?

Choose $P = 1 \Rightarrow \frac{dP}{dt} = 0.75 \cdot \left(1 - \frac{1}{3}\right) \cdot 1 = 0.5 > 0$

the slope is positive so the solution P is increasing for

$0 < P < 3$

(c) Is the solution increasing or decreasing when P is greater than the carrying capacity?

Let $P = 6 \Rightarrow \frac{dP}{dt} = 0.75\left(1 - \frac{6}{3}\right) \cdot 6 = -4.5 < 0$

$\Rightarrow$ Decreasing for $P > 3$.